Recall:

showed:

Lemma: If $\boxed{\sum_{n=0}^{\infty} a_n x^n \text{ has radius of convergence } R}$ general assumptions for this lecture

$\Rightarrow \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$

have radius of convergence R

$\Rightarrow$ **Theorem** (integrate power series)

If $f(x) = \sum a_n x^n$ for $|x| < R$

$\Rightarrow$ Its antiderivative $F(x) = \int_0^x f(t)\,dt$ is given by the power series $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$

Recall: $f_n \to f$ uniformly

does NOT imply in general that $f_n' \to f'$

Situation better for power series.

have not defined differentiability yet

For the moment we say a function F is differentiable if it is the antiderivative of a continuous function

i.e. $F(x) = \int_0^x g(t)\,dt + \underset{\text{const.}}{C}$ for some cont function $g(x)$.

<u>Theorem</u> If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $|x| < R$

$\Rightarrow$ $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$

Proof. By lemma the function $g(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$ has radius of convergence R, i.e. well-defined for $|t| < R$

$$\Rightarrow \quad F(x) = \int_0^x g(t)\, dt$$

$$= \int_0^x \sum_{n=1}^{\infty} n a_n t^{n-1}\, dt$$

previous theorem $\Rightarrow$

$$= \sum_{n=1}^{\infty} \frac{\cancel{n} a_n}{\cancel{n}} t^n$$

$$= \sum_{n=1}^{\infty} a_n t^n$$

$$= f(x) - a_0$$

$\Rightarrow$ $g(t)$ has antiderivatives $F(x)$ and $f(x)$

$\Rightarrow$ $f(x)$ is differentiable with derivative $g(x)$.

Examples:

Recall formula for geometric series

(1) $$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = f(x) \qquad |x| < 1$$

$$\Rightarrow \quad f'(x) = \frac{-1}{(1-x)^2}$$ is given by power series

$$\sum_{n=1}^{\infty} n x^{n-1}$$

$(n-1 \to n)$ $$= \sum_{n=0}^{\infty} (n+1) x^n$$ = power series for $\frac{-1}{(1-x)^2}$

can do same for integration:

$$\int_0^x \frac{1}{1-t}\,dt = -\ln(1-t)\Big|_0^x = -\ln(1-x)$$

is given by power series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}$

shift of exponents $(n+1 \to n)$ $\displaystyle = \sum_{n=1}^{\infty} \frac{1}{n} x^n = -\ln(1-x)$

substitute $x \to -x$

$$\Rightarrow -\ln(1+x) = \sum_{n=1}^{\infty} \frac{1}{n}(-x)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$$

$$\Rightarrow \boxed{\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n} \qquad |x|<1$$

Question: What can we say for $x = \pm R$?

answered at least in part by

Abel's Theorem: $f(x) = \sum a_n x^n$ radius of convergence R

If series converges at $x = R \Rightarrow$ its value is $\lim_{x \to R} f(x)$

" " " " $x = -R \Rightarrow$ " " " $\lim_{x \to -R} f(x)$

Example: we had $\ln(1+x) = \sum \frac{(-1)^{n+1}}{n} x^n$

if $x = 1$ we get the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

This series converges by alternate series test.

Abel's theorem $\Rightarrow$ series converges to $\lim_{x \to 1} \ln(1+x) = \ln(2)$

$$\Rightarrow \quad \boxed{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \ln(2)}$$

⚠ theorem only works if series converges!

<u>Example:</u>

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = f(x)$$

$f(x)$ is continuous for $x = -1$: $f(-1) = \frac{1}{1-(-1)} = \frac{1}{2}$

<u>BUT:</u> $\sum_{n=0}^{\infty} (-1)^n$ does $\overline{\text{NOT}}$ converge!

Proof of Abel's theorem:

Case 1 Assume $R=1$ and series $\sum a_n x^n$ converges at $x=1$

can assume that $\boxed{f(1)=\sum_{n=0}^{\infty} a_k = 0}$

(if not, just subtract constant $f(1)$ from $f(x)$)

$\Rightarrow$ If $S_n = \sum_{k=0}^{n} a_k$

then $\boxed{\lim_{n\to\infty} S_n = 0}$

Want to show: $f_n(x) = \sum_{k=0}^{n} a_k x^k$ converges uniformly to

function $f(x) = \sum_{k=0}^{\infty} a_k x^k$. ($\Rightarrow f$ continuous) $\Rightarrow$ claim

$\Rightarrow$ enough to show that $(f_n(x))_n$ is uniform Cauchy need to estimate (assume $n \geq m$)

$$|f_n(x) - f_m(x)| = \left| \sum_{k=m+1}^{n} a_k x^k \right|$$

$$\boxed{\begin{array}{c} a_k = s_k - s_{k-1} \\ \left(s_k = \sum_{m=0}^{k} a_m \right) \end{array}}$$

$$= \left| \sum_{k=m+1}^{n} (s_k - s_{k-1}) x^k \right|$$

$$= \left| \sum_{k=m+1}^{n} s_k x^k - x \sum_{k=m}^{n-1} s_k x^k \right|$$

shift $k-1 \to k$

$$= \left| \sum_{k=m+1}^{n-1} (1-x) s_k x^k + s_n x^n - s_m x^{m+1} \right|$$

$$\leq \left| \sum_{k=m+1}^{n-1} (1-x) s_k x^k \right| + |s_n x^n| + |s_m x^{m+1}|$$